Lecture 24 : $\mathbf{L}^1$-convergence and reversed MG

STAT205 *Lecturer: Jim Pitman* Scribe: *Alex Skorokhod* <skor@math>

References: [1], sections 4.4, 4.5 and 4.6.

24.1 Martingale Convergence

Recall Dubins' inequality from previous lecture: for a non-negative supermartingale S_n, $n = 0, 1, 2, \ldots$ and $0 < a < b$:

$$\mathbb{P}(S_n \text{ upcrosses } [a,b] \geq k \text{ times}) \leq \left(\frac{a}{b}\right)^k \tag{24.1}$$

As a corollary we get that

Corollary 24.1 $\mathbb{P}(S_n$ *converges to a finite limit*$) = 1$, $S_n \xrightarrow{a.s.} S_\infty$, *and Fatou's lemma yields* $\mathbb{E}(S_\infty) \leq \mathbb{E}(S_0) < \infty$.

We can get a little stronger result ([1], Chapter 4, 2.10) and have the same convergence with non-negativity substituted by $\sup_n \mathbb{E}(S_n^-) < \infty$. Intuitively the condition tells us that we can't go down too much.

24.2 Example

If we look at non-negative integer-valued processes such as random walks or branching processes, then the process must reach zero almost surely. For example consider $S_0 = k > 0$ and $S_n = X_1 + \ldots + X_n$ where X_i are i.i.d with $\mathbb{E}(X) < \infty$. Let $T_0 = \inf\{n : S_n \leq 0\}$ and consider $S_{n \wedge T_0}$. In order to avoid overshooting and guarantee $S_{T_0} = 0$ (on $T_0 < \infty$), assume that $X_i \in \{-1, 0, 1, \ldots\}$. We also want to exclude an always constant process, so assume $\mathbb{P}(X = 0) < 1$. The Strong Law of Large Numbers tells us that $S_n/n \xrightarrow{a.s.} \mathbb{E}(X_1)$. Set $\mathbb{E}(X) = 0$ which makes $S_{n \wedge T_0}$ a non-negative super-martingale, hence it has to converge a.s. On the event $\{T_0 = \infty\}$, however, we must have $X_i \neq 0$ i.o. and hence we can't have convergence on this set. The same holds true for a branching process at critical value.

24.3 More on $\mathbf{L}^2$ Convergence

Consider an L^2-bounded martingale $(M_n, \mathcal{F}_n)$ (that is, with $\sup_n \mathbb{E}(M_n^2) < \infty$). Recall that M_n has orthogonal increments, hence M_n converges in L^2 to some limit M_∞:

$$M_n \xrightarrow{\mathbf{L}^2} M_\infty \text{ and } \mathbb{E}M_\infty^2 = \sup_n \mathbb{E}M_n^2 \tag{24.2}$$

However now we can prove even more - a.s. convergence. There are two ways to prove it. One is to use a version of Kolmogorov's inequality; the other one is to use the martingale convergence theorem and use our control over second moment:

$$\sup_n(\mathbb{E}M_n^-) \le \sup_n \mathbb{E}|M_n| \le \sup_n \sqrt{\mathbb{E}M_n^2} \le \sqrt{\sup_n \mathbb{E}M_n^2} < \infty.$$

Claim 24.2

$$M_n = \mathbb{E}(M_\infty|\mathcal{F}_n) \tag{24.3}$$

In other words, every $\mathbf{L}^2$*-bounded martingale is a sequence of conditional expectations of some target r.v.* M_∞*.*

Proof: We know that $M_n = \mathbb{E}(M_N|\mathcal{F}_n)$ for $N > n$ and we have $M_n \xrightarrow{\mathbf{L}^2} M_\infty$. We want to show that (for $\mathcal{E} = \mathcal{F}_n$),

$$\mathbb{E}(M_n|\mathcal{E}) \xrightarrow{\mathbf{L}^2} \mathbb{E}(M_\infty|\mathcal{E}).$$

The property of $\mathbb{E}(\cdot|\mathcal{E})$ that gives it to us is continuity of $\mathbb{E}(\cdot|\mathcal{E})$ when viewed as an operator on $\mathbf{L}^2$. It's a simple fact from functional analysis. This operator is linear and also bounded since it's norm is 1:

$$E(X|\mathcal{F})^2 \le \mathbb{E}(X|\mathcal{F})^2 + \mathbb{E}(X - E(X|F))^2 = \mathbb{E}(X^2)$$

and 1 is achieved for constant X. ∎

Note that the above result is dominated by Doob's result on martingale convergence which follows from a Kolmogorov-like inequality as mentioned above. The key fact obtained from it is

$$\mathbb{E}\left(\left(\sup_n M_n\right)^2\right) \le 4\sup_n(\mathbb{E}M_n^2)$$

A similar inequality exists for general $\mathbf{L}^p$ for $p > 1$; however for $p = 1$ a constant blows up and analysis gets more difficult.

24.4 Convergence in $\mathbf{L}^1$

Consider a sequence $M_n = \mathbb{E}(M_\infty|\mathcal{F}_n)$ for some $M_\infty \in \mathbf{L}^1$ and filtration $\mathcal{F}_n$. Such a process is a martingale. Let's ask a question of which martingales have this form ("closed in L^1").

Theorem 24.3 *Given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and filtration $\mathcal{F}_n$ the following are equivalent:*

1. *There exists $M_\infty \in L^1$ such that $M_n = \mathbb{E}(M_\infty|\mathcal{F}_n)$;*
2. *$(M_n, \mathcal{F}_n)$ is a martingale which converges in L^1 (and it converges to M_∞); and*
3. *$(M_n, \mathcal{F}_n)$ is a uniformly integrable martingale.*

Let's define uniform integrability, which is somewhat similar to the concept of tightness encountered earlier.

Definition 24.4 *A collection of r.v. $(X_i, i \in I)$ is* uniformly integrable *if*

$$\lim_{x\to\infty} \sup_i \mathbb{E}(|X_i|\mathbf{1}_{|X_i|>x}) = 0.$$

See text ([1], 4.5) for the proof of the following properties of uniform integrability which is the last word on swapping expectations and limits:

Theorem 24.5 *If $X_n \xrightarrow{a.s.} X$ and (X_n) is uniformly integrable then:*

1. *$\mathbb{E}|X| < \infty$; and*
2. *$X_n \xrightarrow{\mathbf{L}^1} X$ and hence $\mathbb{E}(X_n) \to \mathbb{E}(X)$.*

Moreover if $X_n \geq 0$, the converse is true:

$$\mathbb{E}\Big(\lim_n X_n\Big) = \lim \mathbb{E}(X_n) \Leftrightarrow (X_n) \textit{is uniformly integrable.} \tag{24.4}$$

Note that if X_n converges in $\mathbf{L}^1$ it is automatically uniformly integrable.

Proof: (of Thm. 24.3)

1. (1) $\Rightarrow$ (3) Follows from a more general fact (see [1], Chapter 4, (5.1)): Given $(\Omega, \mathcal{F}_0, \mathbb{P})$ and an $X \in \mathbf{L}^1$, the family $\{\mathbb{E}(X|\mathcal{F}) : \mathcal{F} \text{ is a sub-}\sigma\text{-field} \subset \mathcal{F}_0\}$ is uniformly integrable.

2. (3) $\Rightarrow$ (2) Since M_n converges a.s. and is uniformly integrable, it converges in $\mathbf{L}^1$.

3. (2) $\Rightarrow$ (1) Treat the same way as $\mathbf{L}^2$ case and use continuity of $\mathbb{E}(\cdot|\mathcal{F})$ as an operator.

■

To show that uniform integrability is essential, consider simple random walk S_n started at 1 and stopped at first zero ($\mathbb{P}(T_0 < \infty) = 1$). Hence $S_{n\wedge T_0} = 0$ for $n \geq T_0$ and $S_{n\wedge T_0} \xrightarrow{a.s.} 0$. However $\mathbb{E}(S_{n\wedge T_0}) = 1 \neq \mathbb{E}(\lim_{n\to\infty} S_{n\wedge T_0}) = 0$. Hence we can conclude that $S_{n\wedge T_0}$ is not uniformly integrable and does not converge in $\mathbf{L}^1$.

24.5 Reversed Martingales

These arise as $\mathbb{E}(X|\mathcal{G}_n)$ where $\mathcal{G}_n$ is a *decreasing* rather than increasing sequence of $\sigma-$fields. Sometimes they are also called *backwards* martingales.

Example 24.6 *If $\mathcal{G}_n = \sigma(X_n, X_{n+1}, \ldots)$, then $\mathcal{G}_n \downarrow \mathcal{T}(X_1, X_2, \ldots)$.*

In general if $\mathcal{G}_n \downarrow \mathcal{G}_\infty := \bigcap_n \mathcal{G}_n$ then for $X \in \mathbf{L}^1$ the following convergence is true in $\mathbf{L}^1$ and a.s. sense:

$$\mathbb{E}(X|\mathcal{G}_n) \longrightarrow \mathbb{E}(X|\mathcal{G}_\infty). \tag{24.5}$$

The proof is by using the upcrossing inequality.

Definition 24.7 *A sequence of random variables X_i is called* exchangeable *if for all n and any permutation π on n objects we have:*

$$(X_1, \ldots, X_n) \stackrel{d}{=} (X_{\pi(1)}, \ldots, X_{\pi(n)}).$$

The condition of exchangeability is stronger than the assumption of identical distribution of the individual random variables in the sequence, and weaker than the assumption that they are independent and identically distributed.

Example 24.8 *1. All X_i's are i.i.d.*

2. Pick F a random probability distribution on $\mathbb{R}$. Pick X_i as i.i.d. with distribution F.

A famous converse to a previous example is the following theorem:

Theorem 24.9 (de Finetti) *Every exchangeable sequence of real-valued r.v. has the same distribution as some mixture of i.i.d.*

Let $\mathcal{E}_n$ be the σ-field generated by events which are invariant under permutations that leave $n+1, n+2, \ldots$ fixed. Observe that $\mathcal{E}_n \downarrow$ as $n \uparrow$, hence $\mathcal{E}_n \downarrow \mathcal{E}_\infty$ which is called the *exchangeable σ-field*. Note that $\mathcal{E}_\infty$ is richer than the tail-field (in other words, $\mathcal{T}(X_1, X_2, \ldots) \subseteq \mathcal{E}_\infty$).

For example, consider $S_n = X_1 + \ldots + X_n$ for exchangeable X_i. Exchangeability yields

$$\mathbb{E}(X_1|\mathcal{E}_n) = \mathbb{E}(X_2|\mathcal{E}_n) = \ldots = \mathbb{E}(X_n|\mathcal{E}_n)$$

Adding up all terms and dividing by n gives us

$$\frac{\mathbb{E}(S_n|\mathcal{E}_n)}{n} = \mathbb{E}(X_1|\mathcal{E}_n) \Rightarrow \frac{S_n}{n}$$

since S_n is $\mathcal{E}_n$-measurable. Hence we obtain a new proof of the SLLN:

$$\frac{S_n}{n} \xrightarrow{a.s.} \mathbb{E}(X_1|\mathcal{E}_\infty) = \mathbb{E}(X_1)$$

since $\mathcal{E}_\infty$ is trivial by Hewitt-Savage 0-1 law.

References

[1] Richard Durrett. *Probability: theory and examples, 3rd edition.* Thomson Brooks/Cole, 2005.